# APPLICATION OF THE SYMBOLIC METHOD TO THE DERIVATION OF THE EQUATIONS OF THE THEORY OF PLATES

(PRIMENENIE SIMVOLICHESKOGO METODA K VYVODU URAVNENII TEORII PLIT)

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The paper describes a method for deriving the differential equations and boundary conditions for the problems of extension and bending of plates of constant thickness. The method employs the symbolic notation proposed by Lur'e [1] in 1936 for the solutions to the differential equations of the theory of elasticity for a slab. These solutions have the form

$$u = cu_0 - \frac{mz}{2(m-2)} s\partial_1\vartheta_0 + su_0' - \frac{m}{4(m-1)} \lambda\partial_1\vartheta_0'$$
$$v = cv_0 - \frac{mz}{2(m-2)} s\partial_2\vartheta_0 + sv_0' - \frac{m}{4(m-1)} \lambda\partial_2\vartheta_0' \qquad (0.1)$$
$$w = sw_0' + \frac{m}{2(m-2)} \lambda\triangle\vartheta_0 + cw_0 - \frac{mz}{4(m-1)} s\vartheta_0'$$

Here

$$\vartheta_0 = \partial_1 u_0 + \partial_2 v_0 + w_0', \qquad \vartheta_0' = \partial_1 u_0' + \partial_2 v_0' - \triangle w_0$$
$$\triangle = D^2 = \partial_1^2 + \partial_2^2, \qquad \partial_1 = \frac{\partial}{\partial x}, \qquad \partial_2 = \frac{\partial}{\partial y} \qquad (0.2)$$

where $m$ is Poisson's ratio.

The symbols $c$, $s$, $\lambda$ denote the following differential operators

$$c = \cos zD = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} \triangle^n}{(2n)!}, \quad s = \frac{\sin zD}{D} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1} \triangle^n}{(2n+1)!}$$
$$\lambda = \frac{s - zc}{\triangle} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+3} \triangle^n}{(2n+1)!\,(2n+3)} \qquad (0.3)$$

Formulas (0.1) are series in powers of the coordinate $z$ written in a compact form. This symbolic notation is very conveneient in performing the intermediate computations. The quantities $u_0$, $v_0$, $w_0$, $u_0'$, $v_0'$, $w_0'$ which appear here depend on the coordinates $x$ and $y$ and are the fundamental unknowns to be determined. In another paper [2] Lur'e obtained a system of differential equations of infinite order for these functions ($u_0, \ldots, w_0'$); the question of boundary conditions for these, however, was not considered. Obviously, the substitution of the symbolic equations (0.1) into the principle of minimum potential energy must lead both to differential equations and to boundary conditions expressed in the form of series in powers of the thickness of the plate.

An analysis of Formulas (0.1) shows that the displacements of a thick plate fall into two froups distributed symmetrically and skew-symmetrically about the middle plane of the plate. The forst group is characterized by the functions $u_0$, $v_0$, $w_0'$ and corresponds to the extensional problem; the second is defined by the quantities $u_0'$, $v_0'$, $w_0$ and corresponds to the bending problem. The same separation into two groups will exist also in the case of a plate of variable thickness on condition that its middle plane is also a plane of symmetry. This enables us to study each of the above problems separately, which somewhat reduces the volume of computations.

From Formulas (0.1) we can easily derive expressions for the stresses. We give below some of them (where $\mu$ is the shear modulus)

$$
\begin{aligned}
\frac{\sigma_x}{\mu} &= 2c\left(\partial_1 u_0 + \frac{\vartheta_0}{m-2}\right) - \frac{mz}{m-2} s\partial_1^2\vartheta_0 + s\left(2\partial_1 u_0' + \frac{\vartheta_0'}{m-1}\right) - \frac{m\lambda\partial_1^2\vartheta_0'}{2(m-1)}\\
\frac{\sigma_z}{\mu} &= 2c\left(w_0' + \frac{\vartheta_0}{m-2}\right) + \frac{mzs\triangle\vartheta_0}{m-2} - s\left[2\triangle w_0 + \frac{(m-2)\vartheta_0'}{2(m-1)}\right] - \frac{mzc\vartheta_0'}{2(m-1)}\\
\frac{\tau_{xy}}{\mu} &= c(\partial_1 v_0 + \partial_2 u_0) - \frac{mzs\partial_1\partial_2\vartheta_0}{m-2} + s(\partial_1 v_0' + \partial_2 u_0') - \frac{m\lambda\partial_1\partial_2\vartheta_0'}{2(m-1)}\\
\frac{\tau_{zx}}{\mu} &= s(\partial_1 w_0' - \triangle u_0) - \frac{mzc\partial_1\vartheta_0}{m-2} + c(u_0' + \partial_1 w_0') - \frac{mzs\partial_1\vartheta_0'}{2(m-1)}
\end{aligned} \tag{0.4}
$$

The expressions for $\sigma_y$ and $\tau_{yz}$ can be obtained from $\sigma_x$ and $\tau_{zx}$ by replacing $u_0$, $u_0'$, $\partial_1$ by $v_0$, $v_0'$, $\partial_2$.

**1. Variation of potential energy for the extension of a plate.** The variation of the potential energy per unit volume $\delta\pi$ is given by Formula

$$\delta\pi = \sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \sigma_z\delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx} \tag{1.1}$$

In accordance with the foregoing remarks we leave only the functions $u_0$, $v_0$, $w_0'$ in Formulas (0.1) and (0.4) which define extensional deformation. We evaluate the strains from the displacements (0.1) and vary them; then

$$
\begin{aligned}
\delta\varepsilon_x &= c\partial_1\delta u_0 - \frac{mzs}{2(m-2)}\partial_1^2\delta\vartheta_0, \qquad \delta\gamma_{xy} = c(\partial_1\delta v_0 + \partial_2\delta u_0) - \frac{mzs}{2(m-2)}\partial_1\partial_2\delta\vartheta_0\\
\delta\varepsilon_z &= c\delta w_0' + \frac{mzs}{2(m-2)}\triangle\delta\vartheta_0, \qquad \delta\gamma_{zx} = s(\partial_1\delta w_0' - \triangle\delta u_0) - \frac{mzs}{m-2}\partial_1\delta\vartheta_0
\end{aligned} \tag{1.2}
$$

The variations $\delta\varepsilon_y$ and $\delta\gamma_{yz}$ can be obtained from the variations $\delta\varepsilon_x$ and $\delta\gamma_{zx}$ by the appropriate change of letters.

We substitute into Formula (1.1) the stresses (0.4) which depend on the quantities $u_0$, $v_0$, $w_0'$ and the variations of the strains (1.2), we obtain

$$
\begin{aligned}
\frac{\delta\pi_1}{\mu} &= 2(c\partial_1 u_0\cdot c\partial_1\delta u_0 + c\partial_2 v_0\cdot c\partial_2\delta v_0 + cw_0'\cdot c\delta w_0') + \\
&+ c(\partial_1 v_0 + \partial_2 u_0)\cdot c\delta(\partial_1 v_0 + \partial_2 u_0) + \frac{m^2z^2}{(m-2)^2}(c\partial_1\vartheta_0\cdot c\partial_1\delta\vartheta_0 + c\partial_2\vartheta_0\cdot c\partial_2\delta\vartheta_0) + \\
&+ \left\{\frac{2}{m-2}c\vartheta_0\cdot c\delta\vartheta_0\right\} + \frac{mz}{m-2}\delta\,[c\partial_1\vartheta_0\cdot s(\triangle u_0 - \partial_1 w_0') + c\partial_2\vartheta_0\cdot s(\triangle v_0 - \partial_2 w_0') + \\
&+ cw_0'\cdot s\triangle\vartheta_0 - \partial_1 c(u_0\cdot\partial_1 + v_0\cdot\partial_2)s\partial_1\vartheta_0 - \partial_2 c(u_0\cdot\partial_1 + v_0\cdot\partial_2)s\partial_2\vartheta_0] + \\
&+ s(\triangle u_0 - \partial_1 w_0')\cdot s\delta(\triangle u_0 - \partial_1 w_0') + s(\triangle v_0 - \partial_2 w_0')\cdot s\delta(\triangle v_0 - \partial_2 w_0') + \\
&+ \frac{m^2z^2}{2(m-2)^2}(s\partial_1^2\vartheta_0\cdot s\partial_1^2\delta\vartheta_0 + 2s\partial_1\partial_2\vartheta_0\cdot s\partial_1\partial_2\delta\vartheta_0 + s\partial_2^2\vartheta_0\cdot s\partial_2^2\delta\vartheta_0 + s\triangle\vartheta_0\cdot s\triangle\delta\vartheta_0)
\end{aligned} \tag{1.3}
$$

The dots in Formula (1.3) have the following meaning: except for multiplication each dot means the termination of the application of the preceding operator — after the dot the next operator acting on a different function applies.

We intergrate the variation $\delta\pi_1$ over the area of the plate $\Omega$. The result will consist of double integrals containing the variations of the main variables $(\delta u_0, \delta v_0, \delta w_0')$ under the operator symbols. In order to derive the differential equations of the theory of thick plates it is necessary to transform these double integrals in such a way that they include not the operators of the variations but the variations $\delta u_0, \delta v_0, \delta w_0'$ themselves. Such a transformation is made possible by the formula which generalizes the familiar formula of Green to the case of infinite operators [3]. This Formula is

$$\iint\limits_{(\Omega)} [U\cdot\Psi(\triangle)V - \Psi(\triangle)U\cdot V]\,dx\,dy = \sum_{k=1}^{\infty}\oint\limits_{(L)} \{\Psi_k(\triangle)U, V\}\,ds \quad (1.4)$$

Here $L$ is the contour bounding the region $\Omega$, $\Psi_k(\triangle)$ is the $k$th reduced operator from the operator $\Psi(\triangle)$. The process of reducing the operator

$$\Psi(\triangle) = \sum_{r=0}^{\infty} a_r\triangle^r = a_0 + a_1\triangle + a_2\triangle^2 + \ldots$$

consists of discarding the first $k$ terms and at the same time lowering the order of the remaining Laplacians by the same number; thus the $k$th reduced operator is given by the series

$$\Psi_k(\triangle) = \sum_{r=k}^{\infty} a_r\triangle^{r-k} = a_k + a_{k+1}\triangle + a_{k+2}\triangle^2 + \ldots$$

Under the line integral sign in (1.4) there are braces (they could be called Green's braces) which are an abbreviated notation for operations on the pair of functions within the braces. The meaning of this notation is illustrated by Expression

$$\{A_k, B\} = A_k\cdot\frac{\partial\triangle^{k-1}B}{\partial n} - \frac{\partial A_k}{\partial n}\cdot\triangle^{k-1}B \quad (1.5)$$

The transformation of the variation of potential energy by means of Formula (1.4) leads to reduced operators $c_k$, $s_k$, $\lambda_k$ which are defined by the series

$$c_k = \sum_{n=0}^{\infty}\frac{(-1)^{n+k} z^{2n+2k}}{(2n+2k)!}\triangle^n, \qquad s_k = \sum_{n=0}^{\infty}\frac{(-1)^{n+k} z^{2n+2k+1}}{(2n+2k+1)!}\triangle^n \quad (1.6)$$

$$\lambda_k = \sum_{n=0}^{\infty}\frac{(-1)^{n+k} z^{2n+2k+3}\triangle^n}{(2n+2k+1)!\,(2n+2k+3)}$$

After applying the generalized Green's formula (1.4) to the integral $\iint\delta\pi_1 dxdy$, the integrals over the area $\Omega$ will contain only the operators $\partial_1$ and $\partial_2$ of the variations $\delta u_0, \delta v_0, \delta w_0'$. In order to obtain the latter in pure form it will be sufficient to use the usual formulas for integration by parts

$$\iint_{(\Omega)} U\cdot\partial_1 V dx\,dy = \oint_{(L)} U\cdot V n_x ds - \iint_{(\Omega)} \partial_1 U\cdot V dx\,dy$$
$$\iint_{(\Omega)} U\cdot\partial_2 V dx\,dy = \oint_{(L)} U\cdot V n_y ds - \iint_{(\Omega)} \partial_2 U\cdot V dx\,dy \qquad (1.7)$$

Here $n_x$, $n_y$ are the cosines of the angles formed by the normal to the contour $L$ with the axes $x$ and $y$.

As an example, consider the transformation of the term in the braces which appears in the variation of potential energy (1.3); discarding factor $2/(m-2)$, we have, according to Formula (1.4), that

$$\iint_{(\Omega)} c\vartheta_0\cdot c\delta\vartheta_0\,dx\,dy = \iint_{(\Omega)} c^2\vartheta_0\cdot\delta\vartheta_0\,dx\,dy + \sum_{k=1}^{\infty}\oint_{(L)}\{c_k c\vartheta_0,\ \delta\vartheta_0\}\,ds \qquad (1.8)$$

This contains the double integral with the variation $\delta\vartheta_0$, which has still to be transformed by means of Formulas (1.7)

$$\iint_{(\Omega)} c^2\vartheta_0\cdot\delta\vartheta_0\,dx\,dy = \iint_{(\Omega)} c^2\vartheta_2\cdot(\partial_1\delta u_0+\partial_2\delta v_0+\delta w_0')\,dx\,dy =$$
$$= \iint_{(\Omega)} (c^2\vartheta_0\cdot\delta w_0' - c^2\partial_1\vartheta_0\cdot\delta u_0 - c^2\partial_2\vartheta_0\cdot\delta v_0)\,dx\,dy + \oint_{(L)} c^2\vartheta_0\cdot(n_x\delta u_0+n_y\delta v_0)\,ds \qquad (1.9)$$

Substituting the integral (1.9) into (1.8) we obtain finally

$$\iint_{(\Omega)} c\vartheta_0\cdot c\delta\vartheta_0\,dx\,dy = \oint_{(L)} c^2\vartheta_0\cdot\delta u_{n0}\,ds +$$
$$+\sum_{k=1}^{\infty}\oint_{(L)}\{c_k c\vartheta_0,\ \delta\vartheta_0\}\,ds + \iint_{(\Omega)}(c^2\vartheta_0\cdot\delta w_0' - c^2\partial_1\vartheta_0\cdot\delta u_0 - c^2\partial_2\vartheta_0\cdot\delta v_0)\,dx\,dy \qquad (1.10)$$

Here we have introduced the natural notation for the quantity

$$\delta u_{n0} = n_x\delta u_0 + n_y\delta v_0,$$

which is obtained from the variation of the displacement $u_{n0}$ normal to the contour $L$ bounding the middle plane of the plate. The displacements of the middle plane of the plate in directions normal and tangential to the contour $L$ are given by Expressions

$$u_{n0} = n_x u_0 + n_y v_0, \qquad u_{s0} = n_x v_0 - n_y u_0 \qquad (1.11)$$

The variations of the quantities (1.11) will occur in the transformed expression for the variation of the potential enery integrated over the area of the plate.

If we perform analogous transformations to (1.8), (1.9) and (1.10) with the remaining terms in Expression (1.3) (in all Formula (1.3) contains forty such terms), we obtain the variation of the potential extensional strain energy integrated over the area of the plate

$$\frac{1}{\mu}\iint_{(\Omega)}\delta\pi_1\,dx\,dy = \oint_{(L)}\Big(\frac{\partial l^{01}}{\partial n}\cdot\delta u_0 + \frac{\partial l^{02}}{\partial n}\cdot\delta v_0 + l^{00}\cdot\delta w_0' + l^{03}\cdot\delta u_{n0} +$$
$$+ l^{04}\cdot\delta u_{s0} + l^{10}\cdot\delta\vartheta_0 + \frac{\partial l^{11}}{\partial n}\cdot\partial_1\delta\vartheta_0 + \frac{\partial l^{12}}{\partial n}\cdot\partial_2\delta\vartheta_0\Big)\,ds +$$
$$+\sum_{k=1}^{\infty}\oint_{(L)}[\{g_k^{01},\ \delta u_0\} + \{g_k^{02},\ \delta v_0\} + \{g_k^{00},\ \delta w_0'\} + \{g_k^{11},\ \partial_1\delta u_0\} +$$
$$+\{g_k^{12},\ \partial_2\delta v_0\} + \{g_k^{10},\ \delta(\partial_1 v_0+\partial_2 u_0)\} + \{g_k^{21},\ \partial_1\delta w_0'\} + \{g_k^{22},\ \partial_2\delta w_0'\} +$$

(Continued overleaf)

$$+\{g_k^{20},\ \delta\vartheta_0\}+\{g_k^{31},\ \partial_1\delta\vartheta_0\}+\{g_k^{32},\ \partial_2\delta\vartheta_0\}+\{g_k^{41},\ \partial_1^2\delta\vartheta_0\}+\{g_k^{42},\ \partial_2^2\delta\vartheta_0\}+$$

$$+\{[g_k^{40},\ \partial_1\partial_2\delta\vartheta_0\}]\,ds+\iint_{(\Omega)}(l^{(1)}\cdot\delta u_0+l^{(2)}\cdot\delta v_0+l^{(0)}\cdot\delta w_0')\,dx\,dy \tag{1.12}$$

The functions $l^{(j)}$, $l^{ij}$ and $g_k^{ij}$, which appear in the expression for the variation of potential energy (1.12) are defined by Formulas

$$l^{(0)}=2\left(c^2-s^2\triangle+\frac{2mz}{m-2}\,cs\triangle\right)w_0'+\left[s^2\triangle+\frac{2c^2}{m-2}-\frac{m^2z^2}{(m-2)^2}(c^2-s^2\triangle)\ \triangle\right]\vartheta_0$$

$$l^{(1)}=(c^2-s^2\triangle)(\partial_1w_0'-\triangle u_0)-\frac{4mz}{m-2}cs\triangle\partial_1w_0'+ \tag{1.13}$$

$$+\frac{m}{m-2}\left[4zcs\triangle-c^2+\frac{mz^2}{m-2}(c^2-s^2\triangle)\ \triangle\right]\partial_1\vartheta_0$$

$$l^{00}=\frac{\partial s^2w_0'}{\partial n}-n_xs^2\triangle u_0-n_ys^2\triangle v_0-\frac{mz}{m-2}\frac{\partial cs\vartheta_0}{\partial n}$$

$$l^{01}=2c^2u_0-\frac{mz}{m-2}\,cs\partial_1\vartheta_0,\qquad l^{04}=c^2\,(\partial_2u_0-\partial_1u_0)$$

$$l^{03}=\frac{4mz}{m-2}\,cs\triangle w_0'+\frac{2}{m-2}\left[c^2-mzcs\triangle-\frac{m^2z^2}{2(m-2)}(c^2-s^2\triangle)\ \triangle\right]\vartheta_0 \tag{1.14}$$

$$l^{10}=\frac{m^2z^2}{(m-2)^2}\frac{\partial}{\partial n}\left(c^2-\frac{s^2\triangle}{2}\right)\vartheta_0+\frac{2mz}{m-2}\left(n_xcs\triangle u_0+n_ycs\triangle v_0-\frac{1}{2}\frac{\partial csw_0'}{\partial n}\right)$$

$$l^{11}=\frac{m^2z^2}{2(m-2)^2}s^2\partial_1\vartheta_0-\frac{mz}{m-2}\,csu_0$$

$$g_k^{01}=s_{k-1}j^1,\qquad g_k^{00}=2c_kg,\qquad g_k^{11}=2c_k\partial_1g^1,\qquad g_k^{10}=c_k\,(\partial_2g^1+\partial_1g^2)$$

$$g_k^{21}=-s_kj^1,\qquad g_k^{20}=\frac{2c_kc\vartheta_0+mzs_{k-1}g}{m-2},\qquad g_k^{31}=\frac{mz}{m-2}\,c_kj^1 \tag{1.15}$$

$$g_k^{41}=-\frac{mz}{m-2}\,s_k\partial_1g^1,\qquad g_k^{40}=-\frac{mz}{m-2}\,s_k\,(\partial_2g^1+\partial_1g^2)$$

$$g=cw_0'+\frac{mzs\triangle\vartheta_0}{2(m-2)},\qquad g^1=cu_0-\frac{mzs\partial_1\vartheta_0}{2(m-2)}$$

$$j^1=s\,(\triangle u_0-\partial_1w_0')+\frac{mzc\partial_1\vartheta_0}{m-2} \tag{1.16}$$

The formulas for $l^{(2)}$, $l^{02}$, $l^{12}$, $g_k^{02}$, $g_k^{12}$, $g_k^{22}$, $g_k^{32}$, $g_k^{42}$, $g^2$, $j^2$ are obtained from the expressions for $l^{(1)}$, $l^{01}$, $l^{11}$, $g_k^{01}$, $g_k^{11}$, $g_k^{21}$, $g_k^{31}$, $g_k^{41}$, $g^1$, $j^1$ by replacing $u_0$ and $\partial_1$ by $v_0$ and $\partial_2$.

It now remains to integrate Expression (1.12) with respect to the coordinate $z$ (over the thickness of the plate $2h$); in doing so we have to evaluate integrals of the products of the operators $c$, $s$, $\lambda$ and also the reduced operators $c_k$, $s_k$, $\lambda_k$, $s_{k-1}$ etc. The first kind of integrals of the products of trigonometrical functions and polynomials and can be evaluated in the usual manner. We introduce the following notations:

$$C=\cos hD,\qquad S=\frac{\sin hD}{D},\qquad \Lambda=\frac{S-hC}{\triangle} \tag{1.17}$$

The integrals of the first kind can then be expressed in terms of the operators (1.17) and the thickness of the plate; for example

$$\int_{-h}^{h}c^2\,dz=h+CS,\qquad 2\int_{-h}^{h}zcs\,dz=hS^2+C\Lambda=E \tag{1.18}$$

$$\int_{-h}^{h} s^2\,dz = hS^2 - C\Lambda, \qquad \int_{-h}^{h} (c^2 - s^2\triangle)\,dz = 2CS \tag{1.18 cont.}$$

It is more difficult to deal with the integrals which contain products of reduced and ordinary operators. These integrals have to be expressed in series in powers in powers of $h$. Consider, for example, the integral

$$I_k = \int_{-h}^{h} c_k c\,dz = 2\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{n+s+k}h^{2n+2s+2k+1}\triangle^{n+s}}{(2n+2s+2k+1)(2n+2k)!\,(2s)!}$$

Denote $n + s = r$ and sum for the indices $r$ and $s$. If we now group together terms in the appropriate way we obtain

$$I_k = 2\sum_{r=0}^{\infty}\frac{(-1)^{r+k}h^{2r+2k+1}\triangle^r}{2r+2k+1}\sum_{s=0}^{r}\frac{1}{(2r-2s+2k)!\,(2s)!} =$$

$$= \sum_{r=0}^{\infty}(-1)^{r+k}B_{r+k,\,k}^{(0)}\frac{h^{2r+2k+1}}{2r+2k+1}\triangle^r \tag{1.19}$$

Since integrals of the type $I_k$ are summed for the reduction index $k$, we form the sum from Formula (1.19) and thus obtain

$$\sum_{k=1}^{\infty} I_k = \sum_{k=1}^{\infty}\sum_{r=0}^{\infty}(-1)^{r+k}B_{r+k,\,k}^{(0)}\frac{h^{2r+2k+1}}{2r+2k+1}\triangle^r$$

Setting $r + k = p$, summing with respect to $k$ and $p$ and grouping together appropriate terms, we obtain the top line in relations (1.20)

$$\begin{aligned}
\sum_{k=1}^{\infty}\int_{-h}^{h} c_k c\,dz &= \sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+1}}{2p+1}\sum_{k=1}^{p} B_{pk}^{(0)}\triangle^{p-k}\\
\sum_{k=1}^{\infty}\int_{-h}^{h} s_{k-1}s\,dz &= -\sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+1}}{2p+1}\sum_{k=1}^{p} A_{pk}^{(-1)}\triangle^{p-k}\\
\sum_{k=1}^{\infty}\int_{-h}^{h} zs_{k-1}c\,dz &= -\sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+1}}{2p+1}\sum_{k=1}^{p} B_{pk}^{(-1)}\triangle^{p-k}\\
\sum_{k=1}^{\infty}\int_{-h}^{h} s_k s\,dz &= \sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+3}}{2p+3}\sum_{k=1}^{p} A_{pk}^{(1)}\triangle^{p-k}\\
\sum_{k=1}^{\infty}\int_{-h}^{h} zs_k c\,dz &= \sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+3}}{2p+3}\sum_{k=1}^{p} B_{pk}^{(1)}\triangle^{p-k}\\
\sum_{k=1}^{\infty}\int_{-h}^{h} zc_k s\,dz &= \sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+3}}{2p+3}\sum_{k=1}^{p} A_{pk}^{(0)}\triangle^{p-k}\\
\sum_{k=1}^{\infty}\int_{-h}^{h} z^2c_k c\,dz &= \sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+3}}{2p+3}\sum_{k=1}^{p} B_{pk}^{(0)}\triangle^{p-k}\\
\sum_{k=1}^{\infty}\int_{-h}^{h} z^2s_{k-1}s\,dz &= -\sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+3}}{2p+3}\sum_{k=1}^{p} A_{pk}^{(-1)}\triangle^{p-k}\\
\sum_{k=1}^{\infty}\int_{-h}^{h} z^2s_k s\,dz &= \sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+5}}{2p+5}\sum_{k=1}^{p} A_{pk}^{(1)}\triangle^{p-k}
\end{aligned} \tag{1.20}$$

The numbers with three indices $A_{pk}^{(n)}$ and $B_{pk}^{(n)}$, which appear in the operators of (1.20) are defined by Expressions

$$A_{pk}^{(n)} = \sum_{s=0}^{p-k} \frac{2}{(2p-2s+n)!\,(2s+1)!}, \qquad B_{pk}^{(n)} = \sum_{s=0}^{p-k} \frac{2}{(2p-2s+n)!\,(2s)!} \quad (1.21)$$

The values of $A_{pk}^{(n)}$ and $B_{pk}^{(n)}$ for $n=-2, -1, 0, 1, 2, 3$ for certain values of the indices $p$ and $k$ are given in the following tables.

**Table of values of $A_{pk}^{(n)}$**

| $p$ | $k$ | $n=-2$ | $n=-1$ | $n=0$ | $n=1$ | $n=2$ | $n=3$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 2 | 1 | $\frac{1}{3}$ | $\frac{1}{12}$ | $\frac{1}{60}$ |
| 2 | 1 | $\frac{4}{3}$ | $\frac{2}{3}$ | $\frac{1}{4}$ | $\frac{13}{180}$ | $\frac{1}{60}$ | $\frac{1}{315}$ |
| 2 | 2 | 1 | $\frac{1}{3}$ | $\frac{1}{12}$ | $\frac{1}{60}$ | $\frac{1}{360}$ | $\frac{1}{2520}$ |
| 3 | 1 | $\frac{4}{15}$ | $\frac{4}{45}$ | $\frac{1}{40}$ | $\frac{1}{168}$ | $\frac{73}{60480}$ | $\frac{191}{907200}$ |
| 3 | 2 | $\frac{1}{4}$ | $\frac{13}{180}$ | $\frac{1}{60}$ | $\frac{1}{315}$ | $\frac{31}{60480}$ | $\frac{13}{181440}$ |
| 3 | 3 | $\frac{1}{12}$ | $\frac{1}{60}$ | $\frac{1}{360}$ | $\frac{1}{2520}$ | $\frac{1}{20160}$ | $\frac{1}{181440}$ |

**Table of values of $B_{pk}^{(n)}$**

| $p$ | $k$ | $n=-2$ | $n=-1$ | $n=0$ | $n=1$ | $n=2$ | $n=3$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 2 | 1 | $\frac{1}{3}$ | $\frac{1}{12}$ | $\frac{1}{60}$ |
| 2 | 1 | 2 | $\frac{4}{3}$ | $\frac{7}{12}$ | $\frac{11}{60}$ | $\frac{2}{45}$ | $\frac{11}{1260}$ |
| 2 | 2 | 1 | $\frac{1}{3}$ | $\frac{1}{12}$ | $\frac{1}{60}$ | $\frac{1}{360}$ | [illegible]/2520 |
| 3 | 1 | $\frac{2}{3}$ | $\frac{4}{15}$ | $\frac{31}{360}$ | $\frac{19}{840}$ | $\frac{11}{2240}$ | $\frac{163}{181440}$ |
| 3 | 2 | $\frac{7}{12}$ | $\frac{11}{60}$ | $\frac{2}{45}$ | $\frac{11}{1260}$ | $\frac{29}{20160}$ | $\frac{37}{181440}$ |
| 3 | 3 | $\frac{1}{12}$ | $\frac{1}{60}$ | $\frac{1}{360}$ | $\frac{1}{2520}$ | $\frac{1}{20160}$ | $\frac{1}{181440}$ |

We evaluate

$$L^{ij} = \int_{-h}^{h} l^{ij}\,dz$$

using Formulas (1.18) and obtain

$$L^{00} = \frac{\partial}{\partial n}\left[(hS^2 - C\Lambda)\,w_0' - \frac{m}{2(m-2)}\,E\vartheta_0\right] + (n_x CSu_0 + n_y CSv_0) - hu_{n0} \quad (1.22)$$

$$L^{01} = 2(h+CS)\,u_0 - \frac{m}{2(m-2)}\,E\partial_1\vartheta_0, \quad L^{01} = (h+CS)(\partial_2 u_0 - \partial_1 v_0)$$

$$L^{03} = \frac{2m}{m-2}\,E\triangle w_0' + \frac{4hC^2}{m-2}\,\vartheta_0 + \frac{2}{(m-2)^2}\,[2(m-1)E - m^2h^2CS]\,\triangle\vartheta_0$$

$$L^{10} = \frac{m}{m-2}(n_x E\triangle u_0 + n_y E\triangle v_0) + \frac{m}{2(m-2)}\frac{\partial}{\partial n}\left[\frac{mh^2}{m-2}\left(\frac{h}{3}+3CS\right)\vartheta_0 - \right. \qquad \text{(1.22) cont.}$$

$$\left. - E\left(w_0' + \frac{3m\vartheta_0}{2(m-2)}\right)\right]$$

$$L^{11} = \frac{m^2}{4(m-2)^2}\left[\frac{2}{3}h^2(h-3CS)+E\right]\frac{\partial_1\vartheta_0}{\triangle} - \frac{mEu_0}{2(m-2)}$$

Taking into account Formulas (1.17) and (1.18) and introducing the notations

$$H = (m-2)hC^2 - mh^2CS\triangle \tag{1.23}$$

for the operators we can evaluate $L^{(i)} = \int_{-h}^{h} l^{(i)}\,dz$; we obtain

$$L^{(1)} = 2CS(\partial_1 w_0' - \triangle u_0) - \frac{2mE\triangle}{m-2}\partial_1 w_0' - \frac{2m(E\triangle + H)}{(m-2)^2}\partial_1\vartheta_0$$

$$L^{(0)} = 2CS(2w_0' - \vartheta_0) - \frac{2mE\triangle}{m-2}w_0' + \frac{2m[(m-1)E\triangle + H]}{(m-2)^2}(\vartheta_0 + w_0') \tag{1.24}$$

The expressions for $L^{02}, L^{12}$ and $L^{(2)}$ can be obtained from the formulas for $L^{01}, L^{11}$ and $L^{(1)}$ by replacing $u_0$ and $\partial_1$ by $v_0$ and $\partial_2$

Let us now find the variation of the potential energy of the whole plate in its extension

$$\delta\Pi_1 = \int_{-h}^{h} dz \iint_{(\Omega)} \delta\pi_1\,dx\,dy$$

To do so we integrate Expression (1.12) over the thickness of the plate $2h$, making use of the device we have just obtained, namely Formulas (1.20), (1.22) and (1.24). We thus obtain

$$\frac{\delta\Pi_1}{\mu} = \iint_{(\Omega)} (L^{(1)}\cdot\delta u_0 + L^{(2)}\cdot\delta v_0 + L^{(0)}\cdot\delta w_0')\,dx\,dy + \oint_{(L)}\left(\frac{\partial L^{01}}{\partial n}\cdot\delta u_0 + \frac{\partial L^{02}}{\partial n}\cdot\delta v_0 + \right.$$

$$\left. + L^{00}\cdot\delta w_0' + L^{03}\cdot\delta u_{n0} + L^{04}\cdot\delta u_{s0} + L^{10}\cdot\delta\vartheta_0 + \frac{\partial L^{11}}{\partial n}\cdot\delta\partial_1\vartheta_0 + \frac{\partial L^{12}}{\partial n}\cdot\delta\partial_2\vartheta_0\right)ds +$$

$$+ \oint_{(L)}\left\{\sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+1}}{2p+1}\sum_{k=1}^{p}\left[A_{pk}^{(-1)}\Phi_{pk}^{1} + 2B_{pk}^{(0)}\Phi_{pk}^{2} - \frac{m}{m-2}B_{pk}^{(-1)}\Phi_{pk}^{3}\right] + \right.$$

$$+ \sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+3}}{2p+3}\sum_{k=1}^{p}\left[A_{pk}^{(1)}\Phi_{pk}^{4} + \frac{m}{m-2}(A_{pk}^{(0)}\Phi_{pk}^{5} - B_{pk}^{(1)}\Phi_{pk}^{6}) + \right. \tag{1.25}$$

$$\left.\left. + \frac{m^2}{(m-2)^2}(B_{pk}^{(0)}\Phi_{pk}^{7} - A_{pk}^{(-1)}\Phi_{pk}^{8})\right] + \frac{m^2}{(m-2)^2}\sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+5}}{2p+5}\sum_{k=1}^{p}A_{pk}^{(1)}\Phi_{pk}^{9}\right\}ds$$

Here the quantities $\Phi_{pk}^{j}$ can be expressed by means of modified Green's braces

$$[A, B] = \triangle^{p-k}A\cdot\frac{\partial\triangle^{k-1}B}{\partial n} - \frac{\partial\triangle^{p-k}A}{\partial n}\cdot\triangle^{k-1}B \tag{1.26}$$

Thus $\Phi_{pk}^{j}$ can be expressed by means of the braces

$$\Phi_{pk}^{1} = [\partial_1 w_0' - \triangle u_0, \delta u_0] + [\partial_2 w_0' - \triangle v_0, \delta v_0], \quad \Phi_{pk}^{7} = [\partial_1\vartheta_0, \partial_1\delta\vartheta_0] + [\partial_2\vartheta_0, \partial_2\delta\vartheta_0]$$

$$\Phi_{pk}^{2} = [\partial_1 u_0, \partial_1\delta u_0] + {}^1/_2\,[(\partial_1 v_0 + \partial_2 u_0), \delta(\partial_1 v_0 + \partial_2 u_0)] + [\partial_2 v_0, \partial_2\delta v_0] + [w_0', \delta w_0'] +$$

$$+ \frac{1}{m-2}[\vartheta_0, \delta\vartheta_0], \quad \Phi_{pk}^{3} = [\partial_1\vartheta_0, \delta u_0] + [\partial_2\vartheta_0, \delta v_0] + [w_0', \delta\vartheta_0]$$

$$\Phi_{pk}^{4}=[\partial_1 w_0'-\triangle u_0,\ \partial_1\delta w_0']+[\partial_2 w_0'-\triangle v_0,\ \partial_2\delta w_0'],\qquad \Phi_{pk}^{8}={}^1/_2\,[\triangle\vartheta_0,\ \delta\vartheta_0]$$
$$\Phi_{pk}^{5}=[\triangle u_0-\partial_1 w_0',\ \partial_1\delta\vartheta_0]+[\triangle v_0-\partial_2 w_0',\ \partial_2\delta\vartheta_0]+[\triangle\vartheta_0,\ \delta w_0']-[\partial_1^2\vartheta_0,\ \partial_1\delta u_0]-$$
$$-[\partial_1\partial_2\vartheta_0,\ \delta(\partial_1 v_0+\partial_2 u_0)]-[\partial_2^2\vartheta_0,\ \partial_2\delta v_0],\qquad \Phi_{pk}^{6}=[\partial_1\vartheta_0,\ \delta\partial_1 w_0']+$$
$$+[\partial_2\vartheta_0,\ \partial_2\delta w_0']+[\partial_1 u_0,\ \partial_1^2\delta\vartheta_0]+[\partial_1 v_0+\partial_2 u_0,\ \partial_1\partial_2\delta\vartheta_0]+[\partial_2 v_0,\ \partial_2^2\delta\vartheta_0]$$
$$\Phi_{pk}^{9}=[\partial_1^2\vartheta_0,\ \partial_1^2\delta\vartheta_0]+2[\partial_1\partial_2\vartheta_0,\ \partial_1\partial_2\delta\vartheta_0]+[\partial_2^2\vartheta_0,\ \partial_2^2\delta\vartheta_0] \tag{1.27}$$

## 2. Variation of potential energy of bending of a plate.

The deformation of a plate in bending is characterized by the quantities $u_0'$, $v_0'$, $w_0$. Evaluating the variation of the deformations corresponding to bending in accordance with (0.1), we obtain

$$\delta\varepsilon_x=s\partial_1\delta u_0'-\frac{m\lambda\partial_1^2}{4(m-1)}\delta\vartheta_0',\quad \delta\gamma_{xy}=s(\partial_1\delta v_0'+\partial_2\delta u_0')-\frac{m\lambda\partial_1\partial_2}{2(m-1)}\delta\vartheta_0' \tag{2.1}$$
$$\delta\varepsilon_z=-s\triangle\delta w_0-\frac{m(s+zc)}{4(m-1)}\delta\vartheta_0',\qquad \delta\gamma_{zx}=c(\delta u_0'+\partial_1\delta w_0)-\frac{mzs}{2(m-1)}\partial_1\delta\vartheta_0'$$

The variations $\delta\varepsilon_y$ and $\delta\gamma_{yz}$ can be obtained from the variations $\delta\varepsilon_x$ and $\delta\gamma_{zx}$ by an appropriate change of letters. We substitute into (1.1) the bending stresses corresponding to $u_0'$, $v_0'$, $w_0$ from Formulas (0.4) and the variation of deformations (2.1). Then

$$\frac{\delta\pi_2}{\mu}=2(s\partial_1 u_0'\cdot s\partial_1\delta u_0'+s\partial_2 v_0'\cdot s\partial_2\delta v_0'+s\triangle w_0\cdot s\triangle\delta w_0)+\frac{m\delta(s\triangle w_0\cdot s\vartheta_0')}{2(m-1)}+$$
$$+s(\partial_1 v_0'+\partial_2 u_0')\cdot s\delta(\partial_1 v_0'+\partial_2 u_0')+\frac{m^2+6m-8}{8(m-1)^2}s\vartheta_0'\cdot s\delta\vartheta_0'+$$
$$+\frac{m^2z^2}{4(m-1)^2}(s\partial_1\vartheta_0'\cdot s\partial_1\delta\vartheta_0'+s\partial_2\vartheta_0'\cdot s\partial_2\delta\vartheta_0')+\frac{m(m-2)z}{8(m-1)^2}s\vartheta_0'\cdot c\delta\vartheta_0'-$$
$$-\frac{mz}{2(m-1)}\delta[c(u_0'+\partial_1 w_0)\cdot s\partial_1\vartheta_0'+c(v_0'+\partial_2 w_0)\cdot s\partial_2\vartheta_0'-c\vartheta_0'\cdot s\triangle w_0]+$$
$$+c(u_0'+\partial_1 w_0)\cdot c\delta(u_0'+\partial_1 w_0)+c(v_0'+\partial_2 w_0)\cdot c\delta(v_0'+\partial_2 w_0)+$$
$$+\frac{m^2zc\vartheta_0'}{8(m-1)^2}\cdot(s+zc)\delta\vartheta_0'-\frac{m}{2(m-1)}\delta[\partial_1 s(u_0'\cdot\partial_1+v_0'\cdot\partial_2)\partial_1\lambda\vartheta_0'+$$
$$+\partial_2 s(u_0'\cdot\partial_1+v_0'\cdot\partial_2)\partial_2\lambda\vartheta_0']-\frac{ms\vartheta_0'}{4(m-1)^2}\cdot\lambda\triangle\delta\vartheta_0'+ \tag{2.2}$$
$$+\frac{m^2}{8(m-1)^2}(\lambda\partial_1^2\vartheta_0'\cdot\lambda\partial_1^2\delta\vartheta_0'+2\lambda\partial_1\partial_2\vartheta_0'\cdot\lambda\partial_1\partial_2\delta\vartheta_0'+\lambda\partial_2^2\vartheta_0'\cdot\lambda\partial_2^2\delta\vartheta_0')$$

Integrating the variation of potential energy per unit volume (2.2) over the area of the plate $\Omega$ and transforming the integrals so obtained (in all there will be forty six) by means of the generalized Green's formula (1.4) and the formulas for integration by parts (1.7), we finally obtain

$$\frac{1}{\mu}\iint\limits_{(\Omega)}\delta\pi_2\,dx\,dy=\oint\limits_{(L)}\Big(\frac{\partial k^{01}}{\partial n}\cdot\delta u_0'+\frac{\partial k^{02}}{\partial n}\cdot\delta v_0'+k^{00}\cdot\delta w_0+k^{03}\cdot\delta\omega_0+$$
$$+k^{04}\cdot\delta u_{s0}'+k^{10}\cdot\delta\vartheta_0'+\frac{\partial k^{11}}{\partial n}\cdot\delta\partial_1\vartheta_0'+\frac{\partial k^{12}}{\partial n}\cdot\delta\partial_2\vartheta_0'\Big)ds+$$
$$+\sum_{k=1}^{\infty}\oint\limits_{(L)}[\{f_k^{01},\ \delta(u_0'+\partial_1 w_0)\}+\{f_k^{02},\ \delta(v_0'+\partial_2 w_0)\}+\{f_k^{00},\ \delta w_0\}+\{f_k^{11},\ \partial_1\delta u_0'\}+$$

$$+\{f_k^{12}, \partial_2\delta v_0'\}+\{f_k^{10}, \delta(\partial_1 v_0'+\partial_2 u_0')\}+\{f_k^{20}, \delta\vartheta_0'\}+\{f_k^{31}, \partial_1\delta\vartheta_0'\}+\{f_k^{32}, \partial_2\delta\vartheta_0'\}+$$

$$+\{f_k^{41}, \partial_1^2\delta\vartheta_0'\}+\{f_k^{42}, \partial_2^2\delta\vartheta_0'\}+\{f_k^{40}, \partial_1\partial_2\delta\vartheta_0'\}]\,ds+$$

$$+\iint\limits_{(\Omega)}(k^{(1)}\cdot\delta u_0'+k^{(2)}\cdot\delta v_0'+k^{(0)}\cdot\delta w_0)\,dx\,dy \tag{2.3}$$

Relations (2.3) contain the variations of the quantities

$$\omega_0=n_x u_0'+n_y v_0'-\frac{\partial w_0}{\partial n}, \qquad u_{s0}'=n_x v_0'-n_y u_0' \tag{2.4}$$

The functions $k^{(j)}$, $k^{ij}$ and $f_k^{ij}$, which appear in the expression for the variation of potential energy (2.3) are determined by Formulas

$$k^{(1)}=(c^2-s^2\triangle)(u_0'+\partial_1 w_0)-\frac{2mz}{m-1}cs\triangle\partial_1 w_0- \tag{2.5}$$

$$-\frac{m}{4(m-1)^2}[(m-2)s^2+8(m-1)zcs+mz^2(c^2-s^2\triangle)]\ \partial_1\vartheta_0'$$

$$k^{(0)}=-2\left(c^2-s^2\triangle+\frac{mz}{m-1}cs\triangle\right)\triangle w_0+$$

$$+\frac{1}{4(m-1)^2}[(3m^2-6m+4)s^2\triangle-4(m-1)^2c^2-m^2z^2(c^2-s^2\triangle)\triangle]\,\vartheta_0'$$

$$k^{00}=n_x c^2 u_0'+n_y c^2 v_0'+\frac{\partial}{\partial n}\left[c^2 w_0-\frac{mzcs\vartheta_0'}{2(m-1)}+k^{03}\right], \qquad k^{01}=2s^2u_0'-\frac{m\lambda s\partial_1\vartheta_0'}{2(m-1)}$$

$$k^{03}=\frac{2mzcs\triangle w_0}{m-1}+\frac{1}{4(m-1)^2}[4m(m-1)zcs-(m-2)^2s^2+m^2z^2(c^2-s^2\triangle)]\,\vartheta_0'$$

$$k^{04}=s^2(\partial_2 u_0'-\partial_1 v_0'), \qquad k^{11}=\frac{m\lambda}{2(m-1)}\left[\frac{m\lambda\partial_1\vartheta_0'}{4(m-1)}-su_0'\right] \tag{2.6}$$

$$k^{10}=\frac{m}{2(m-1)}[n_x(\lambda\triangle-zc)su_0'+n_y(\lambda\triangle-zc)sv_0']-\frac{mz}{2(m-1)}\frac{\partial csw_0}{\partial n}+$$

$$+\frac{m^2}{4(m-1)^2}\frac{\partial}{\partial n}\left(z^2s^2-\frac{\lambda^2\triangle}{2}\right)\vartheta_0'$$

$$f_k^{01}=c_k\varphi^1, \quad f_k^{00}=2s_{k-1}f, \quad f_k^{11}=2s_{k-1}\partial_1 f^1, \quad f_k^{10}=s_k(\partial_2 f^1+\partial_1 f^2)$$

$$f_k^{20}=\frac{m}{2(m-1)}(s_k+zc_k)f+\frac{m-2}{2(m-1)^2}s_k s\vartheta_0', \quad f_k^{31}=-\frac{mzs_k\varphi^1}{2(m-1)} \tag{2.7}$$

$$f_k^{41}=-\frac{m\lambda_k\partial_1 f^1}{2(m-1)}, \qquad f_k^{40}=-\frac{m\lambda_k}{2(m-1)}(\partial_2 f^1+\partial_1 f^2)$$

Here (2.8)

$$f=s\triangle w_0+\frac{m(s+zc)}{4(m-1)}\vartheta_0', \quad f^1=su_0'-\frac{m\lambda\partial_1\vartheta_0'}{4(m-1)}, \quad \varphi^1=c(u_0'+\partial_1 w_0)-\frac{mzs\partial_1\vartheta_0'}{2(m-1)}$$

Formulas for $k^{(2)}$, $k^{02}$, $k^{12}$, $f_k^{02}$, $f_k^{12}$, $f_k^{32}$, $f_k^{42}$, $f^2$ and $\varphi^2$ can be obtained from (2.5) to (2.8) by changing the letters.

In the integration of the Expressions (2.3) over the thickness of the plate, in addition to Formulas (1.20) we must use analogous relations containing the operators $\lambda$ and $\lambda_k$ for the subsequent transformations. These relations can be expressed in the form

$$\sum_{k=1}^{\infty}\int_{-h}^{h}s_k\lambda\,dz=\sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+5}}{2p+5}\sum_{k=1}^{p}C_{pk}\triangle^{p-k}$$

$$\sum_{k=1}^{\infty}\int_{-h}^{h}\lambda_k s dz=\sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+5}}{2p+5}\sum_{k=1}^{p}D_{pk}\triangle^{p-k}$$
$$\sum_{k=1}^{\infty}\int_{-h}^{h}\lambda_k \lambda dz=\sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+7}}{2p+7}\sum_{k=1}^{p}F_{pk}\triangle^{p-k} \tag{2.9}$$

The derivation of relations (2.9) follows the same pattern as (1.20). The coefficients $C_{pk}$, $D_{pk}$ and $F_{pk}$ can be expressed in terms of the numbers with three indices (1.21)

$$C_{pk}=B_{p+1,\,k}^{(1)}-A_{p+1,\,k}^{(1)},\qquad E_{pk}=A_{pk}^{(0)}+A_{pk}^{(1)}+B_{pk}^{(0)}+B_{pk}^{(1)}$$
$$D_{pk}=A_{pk}^{(2)}-A_{pk}^{(3)},\qquad F_{pk}=B_{p+1,\,k}^{(2)}-A_{p+1,\,k}^{(2)}-B_{p+1,\,k}^{(3)}+A_{p+1,\,k}^{(3)} \tag{2.10}$$

We now evaluate the operators

$$K^{ij}=\int_{-h}^{h}k^{ij}dz,\qquad K^{(j)}=\int_{-h}^{h}k^{(j)}dz$$

using integrals of the type (1.18). We obtain

$$K^{01}=2(hS^2-C\Lambda)u_0'-\frac{m(hS^2-3C\Lambda)}{4(m-1)\triangle}\partial_1\vartheta_0',\quad K^{04}=(hS^2-C\Lambda)(\partial_2u_0'-\partial_1v_0')$$
$$K^{00}=h\left(n_xu_0'+n_yv_0'+\frac{\partial w_0}{\partial n}\right)+n_xCSu_0'+n_yCSv_0'+$$
$$+\frac{\partial}{\partial n}\left[CSw_0-\frac{mE\vartheta_0'}{4(m-1)}+K^{03}\right]$$
$$K^{03}=\frac{mE\triangle w_0}{m-1}+\frac{1}{2(m-1)^2}[(m-1)(m-2)E-(m-2)^2hS^2+m^2h^2CS]\vartheta_0'$$
$$K^{10}=-\frac{m}{4(m-1)}\frac{\partial Ew_0}{\partial n}-\frac{m}{m-1}(n_xC\Lambda u_0'+n_yC\Lambda v_0')+$$
$$+\frac{m^2}{8(m-1)^2}\frac{\partial}{\partial n}\left(\frac{h^3}{3}+\frac{7}{2}E-2hS^2-3h^2CS\right)\frac{\vartheta_0'}{\triangle}$$
$$K^{11}=\frac{m^2}{8(m-1)^2}\left(\frac{h^2}{3}-\frac{5}{2}E+2hS^2+h^2CS\right)\frac{\partial_1\vartheta_0'}{\triangle^2}-\frac{m(hS^2-3C\Lambda)}{4(m-1)\triangle}u_0' \tag{2.11}$$

$$K^{(1)}=2CS(u_0'+\partial_1w_0)-\frac{mE\triangle}{m-1}\partial_1w_0-\frac{m(E+F)}{2(m-1)}\partial_1\vartheta_0'$$
$$K^{(0)}=-2CS(\vartheta_0'+2\triangle w_0)-\frac{mE\triangle^2}{m-1}w_0-\frac{m(E-F)}{2(m-1)}\triangle\vartheta_0' \tag{2.12}$$
$$F=\frac{mh^2CS+(m-2)hS^2}{m-1} \tag{2.13}$$

Formulas for the quantities $K^{02}$, $K^{12}$, $K^{(2)}$ can be obtained from the formulas for $K^{01}$, $K^{11}$, $K^{(1)}$ by replacing $u_0'$ and $\partial_1$ by $v_0'$ and $\partial_2$.
The variation of the potential bending energy of the whole plate in its flexure is obtained by integrating expression (2.3) over the thickness. We have

$$\frac{\delta\Pi_2}{\mu} = \iint\limits_{(\Omega)} (K^{(1)}\cdot\delta u_0' + K^{(2)}\cdot\delta v_0' + K^{(0)}\cdot\delta w_0)\,dx\,dy +$$

$$+\oint\limits_{(L)}\Big(\frac{\partial K^{01}}{\partial n}\cdot\delta u_0' + \frac{\partial K^{02}}{\partial n}\cdot\delta v_0' + K^{00}\cdot\delta w_0 + K^{03}\cdot\delta\omega_0 + K^{04}\cdot\delta u_{s0}' + K^{10}\cdot\delta\vartheta_0' +$$

$$+\frac{\partial K^{11}}{\partial n}\cdot\delta\partial_1\vartheta_0' + \frac{\partial K^{12}}{\partial n}\cdot\delta\partial_2\vartheta_0'\Big)ds + \oint\limits_{(L)}\Big\{\sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+1}}{2p+1}\sum_{k=1}^{p}\Big[B_{pk}^{(0)}\Psi_{pk}^{1} -$$

$$-2A_{pk}^{(-1)}\Psi_{pk}^{2} - \frac{m}{2(m-1)}(A_{pk}^{(-1)} + B_{pk}^{(-1)})\Psi_{pk}^{3}\Big] + \sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+3}}{2p+3}\times$$

$$\times\sum_{k=1}^{p}\Big[A_{pk}^{(1)}\Psi_{pk}^{4} + \frac{m}{2(m-1)}(A_{pk}^{(0)}\Psi_{pk}^{5} - B_{pk}^{(1)}\Psi_{pk}^{6}) + \frac{m^2E_{pk}}{8(m-1)^2}\Psi_{pk}^{7}\Big] +$$

$$+\frac{m}{2(m-1)}\sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+5}}{2p+5}\sum_{k=1}^{p}\Big[\frac{m}{2(m-1)}A_{pk}^{(1)}\Psi_{pk}^{8} - C_{pk}\Psi_{pk}^{9} - D_{pk}\Psi_{pk}^{10}\Big] +$$

$$+\frac{m^2}{8(m-1)^2}\sum_{p=1}^{\infty}\frac{(-1)^p h^{2p+7}}{2p+7}\sum_{k=1}^{p}F_{pk}\Psi_{pk}^{11}\Big\}ds \tag{2.14}$$

The functions $\Psi_{pk}^{j}$ are defined using the modified Green's braces (1.26) by the following relations:

$$(2.15)$$

$$\Psi_{pk}^{0} = [\triangle w_0,\ \delta\vartheta_0'], \qquad \Psi_{pk}^{1} = [u_0' + \partial_1 w_0,\ \delta(u_0' + \partial_1 w_0)] + [v_0' + \partial_2 w_0,\ \delta(v_0' + \partial_2 w_0)]$$

$$\Psi_{pk}^{2} = [\triangle w_0,\ \delta w_0], \qquad \Psi_{pk}^{3} = [\vartheta_0',\ \delta w_0], \qquad \Psi_{pk}^{7} = [\vartheta_0',\ \delta\vartheta_0']$$

$$\Psi_{pk}^{4} = 2[\partial_1 u_0',\ \delta\partial_1 u_0'] + [\partial_1 v_0' + \partial_2 u_0',\ \delta(\partial_1 v_0' + \partial_2 u_0')] + 2[\partial_2 v_0',\ \delta\partial_2 v_0'] + \frac{m}{2(m-1)}\Psi_{pk}^{0} + \frac{m-2}{2(m-1)^2}\Psi_{pk}^{7}, \quad \Psi_{pk}^{8} = [\partial_1\vartheta_0',\ \delta\partial_1\vartheta_0'] + [\partial_2\vartheta_0',\ \delta\partial_2\vartheta_0']$$

$$\Psi_{pk}^{5} = \Psi_{pk}^{0} - [\partial_1\vartheta_0',\ \delta(u_0' + \partial_1 w_0)] - [\partial_2\vartheta_0',\ \delta(v_0' + \partial_2 w_0)]$$

$$\Psi_{pk}^{6} = [u_0' + \partial_1 w_0,\ \delta\partial_1\vartheta_0'] + [v_0' + \partial_2 w_0,\ \delta\partial_2\vartheta_0']$$

$$\Psi_{pk}^{11} = [\partial_1^2\vartheta_0',\ \delta\partial_1^2\vartheta_0'] + 2[\partial_1\partial_2\vartheta_0',\ \delta\partial_1\partial_2\vartheta_0'] + [\partial_2^2\vartheta_0',\ \delta\partial_2^2\vartheta_0']$$

$$\Psi_{pk}^{9} = [\partial_1^2\vartheta_0',\ \delta\partial_1 u_0'] + [\partial_1\partial_2\vartheta_0',\ \delta(\partial_2 u_0' + \partial_1 v_0')] + [\partial_2^2\vartheta_0',\ \delta\partial_2 v_0']$$

$$\Psi_{pk}^{10} = [\partial_1 u_0',\ \delta\partial_1^2\vartheta_0'] + [\partial_2 u_0' + \partial_1 v_0',\ \delta\partial_1\partial_2\vartheta_0'] + [\partial_2 v_0',\ \delta\partial_2^2\vartheta_0']$$

**3. Elemental work done by the external forces applied to the plate.** We first calculate the work done by the forces applied to the faces of the plate; the elemental work of these forces is

$$\delta A' = \iint\limits_{(\Omega)} (\mathbf{p}^+\cdot\delta\mathbf{u}^+ + \mathbf{p}^-\cdot\delta\mathbf{u}^-)\,dx\,dy \tag{3.1}$$

Here $\mathbf{p}^+$ denotes the vector of the external forces per unit area of the face $z = h$; the vector $\mathbf{p}^-$ acts on the face $z = -h$; $\mathbf{u}^+$ and $\mathbf{u}^-$ are vectors of the displacements of points on the faces $z = \pm h$. Expanding the scalar products in (3.1), we obtain

$$\mathbf{p}^+\cdot\delta\mathbf{u}^+ + \mathbf{p}^-\cdot\delta\mathbf{u}^- = p_x^+\delta u^+ + p_y^+\delta v^+ + p_z^+\delta w^+ + p_x^-\delta u^- + p_y^-\delta v^- + p_z^-\delta w^-$$

The values of the variations of the displacements at the faces can easily be obtained by varying Formulas (0.1) and substituting for $\epsilon$ the values $+h$ or $-h$; for example,

$$\delta u_{\pm}^{+} = C\delta u_0 - \frac{mhS}{2(m-2)}\partial_1\delta\vartheta_0 + S\delta u_0' - \frac{m\Lambda}{4(m-1)}\partial_1\delta\vartheta_0' \tag{3.2}$$

The operators $C$, $S$, $\Lambda$ which appear in Formula (3.2) have already been defined by (1.17).

It is evident that the problem of the extension of a plate corresponds to combinations of surface loadings given by Formulas

$$p_x^+ + p_x^- = \eta_x, \qquad p_y^+ + p_y^- = \eta_y, \qquad p_z^+ - p_z^- = \zeta \tag{3.3}$$

whereas the bending problem corresponds to

$$p_x^+ - p_x^- = t_x, \qquad p_y^+ - p_y^- = t_y, \qquad p_z^+ + p_z^- = p \tag{3.4}$$

In addition, we introduce the differential combinations of the loading

$$\partial_1\eta_x + \partial_2\eta_y = \eta^*, \qquad \partial_1 t_x + \partial_2 t_y = t^* \tag{3.5}$$

Then, substituting the variations of the surface displacements (formulas of the type (3.2)) into the elemental work (3.1) and taking into account the notations (3.3) and (3.4), we obtain separate expressions for the elemental work done by the surface forces in the problem of extension ($\delta A_1$) and in the problem of bending ($\delta A_2$) of a thick plate

$$\delta A_1 = \iint_{(\Omega)} \Big[\eta_x\cdot C\delta u_0 + \eta_y\cdot C\delta v_0 + \zeta\cdot S\delta w_0' - \\ - \frac{m}{2(m-2)}\,(\eta_x\cdot hS\partial_1 + \eta_y\cdot hS\partial_2 - \zeta\cdot\Lambda\triangle)\,\delta\vartheta_0\Big]\,dx\,dy \tag{3.6}$$

$$\delta A_2 = \iint_{(\Omega)} \Big[t_x\cdot S\delta u_0' + t_y\cdot S\delta v_0' + p\cdot C\delta w_0 - \\ - \frac{m}{4(m-1)}\,(t_x\cdot\Lambda\partial_1 + t_y\cdot\Lambda\partial_2 + p\cdot hS)\,\delta\vartheta_0'\Big]\,dx\,dy$$

We transform Expressions (3.6) with the aid of (1.4) and Formulas (1.7). Using the notations (1.5),(1.11),(2.4) and (3.5), we omit the calculations and simply state the result

$$\delta A_1 = \iint_{(\Omega)} [(C\eta_x - \delta_1\Xi)\cdot\delta u_0 + (C\eta_y - \partial_2\Xi)\cdot\delta v_0 + (S\zeta + \Xi)\cdot\delta w_0']\,dx\,dy + \\ + \sum_{k=1}^{\infty}\oint_{(L)} [\{C_k\eta_x,\ \delta u_0\} + \{C_k\eta_y,\ \delta v_0\} + \{S_k\zeta,\ \delta w_0'\} + \\ + \frac{m}{2(m-2)}(\{\Lambda_{k-1}\,\zeta,\ \delta\vartheta_0\} - h\{S_k\eta_x,\ \delta\partial_1\vartheta_0\} - h\{S_k\eta_y,\ \delta\partial_2\vartheta_0\})]\,ds + \\ + \oint_{(L)}\Big[\Xi\cdot\delta u_{n0} - \frac{mh}{2(m-2)}(n_x S\eta_x + n_y S\eta_y)\cdot\delta\vartheta_0\Big]\,ds \tag{3.7}$$

$$\delta A_2 = \iint_{(\Omega)} [(St_x - \partial_1\Theta)\cdot\delta u_0' + (St_y - \delta_2\Theta)\cdot\delta v_0' + (Cp - \triangle\Theta)\cdot\delta w_0]\,dx\,dy + \\ + \sum_{k=1}^{\infty}\oint_{(L)} \Big[\{S_k t_x,\ \delta u_0'\} + \{S_k t_y,\ \delta v_0'\} + \{C_k p,\ \delta w_0\} - \\ - \frac{m}{4(m-1)}(\{\Lambda_k t_x,\ \delta\partial_1\vartheta_0'\} + \{\Lambda_k t_y,\ \delta\partial_2\vartheta_0'\} + \{hS_k p,\ \delta\vartheta_0'\})\Big]\,ds + \\ + \oint_{(L)}\Big[\Theta\delta\omega_0 + \frac{\partial\Theta}{\partial n}\cdot\delta w_0 - \frac{m}{4(m-1)}(n_x\Lambda t_x + n_y\Lambda t_y)\cdot\delta\vartheta_0'\Big]\,ds \tag{3.8}$$

Here for brevity we have introduced the quantities $\Xi$ and $\Theta$, which denote the following differential operations on the surface loading:

$$\Xi = \frac{m}{2(m-2)}(hS\eta^* + \Lambda\triangle\zeta), \qquad \Theta = \frac{m}{4(m-1)}(\Lambda t^* - hSp) \tag{3.9}$$

Note that without loss of generality we can assume that there is no surface loading $\mathbf{p}$, since the problem of the equilibrium of a thick plate (both in bending and in extension) can always be assumed to consist of two parts: (1) the problem of the equilibrium of an infinite slab under the action of surface loading, the solution of which is known [4], and (2) the problem of the equilibrium of a thick plate loaded at the edges with no loading on the faces. Thus the problem of the state of stress in a thick plate is reduced in fact to the problem of finding homogeneous solutions which correspond to the absence of loading on the faces [5]. The derivation of Formulas (3.7) and (3.8) was carried out with the aim purely to give a uniform approach to loadings of any kind.

We proceed now to the calculation of the work done by the external forces applied to the faces of the plate. We denote the vector of the surface loading per unit area by $\mathbf{q}_n$; then the elemental work done by the external forces will be

$$\delta A'' = \int_{-h}^{h} dz \oint_{(L)} \mathbf{q}_n \cdot \delta\mathbf{u}\, ds = \int_{-h}^{h} dz \oint_{(L)} (q_{nx}\delta u + q_{ny}\delta v + q_{nz}\delta w)\, ds \tag{3.10}$$

We vary $u, v, w$ and substitute the result into (3.10) in accordance with Formulas (0.1). Then we change the order of integration and separate the parts of the elemental work done by the external forces which refer to extension $\delta A_3$ and bending $\delta A_4$. We obtain

$$\delta A_3 = \oint_{(L)} \Big\{ \int_{-h}^{h} q_{nx} c\,dz\, \delta u_0 + \int_{-h}^{h} q_{ny} c\,dz\, \delta v_0 + \int_{-h}^{h} q_{nz} s\,dz\, \delta w_0' -$$

$$- \frac{m}{2(m-2)} \Big( \int_{-h}^{h} q_{nx} sz\, dz \delta\partial_1\vartheta_0 + \int_{-h}^{h} q_{ny} sz\, dz\delta\partial_2\vartheta_0 - \int_{-h}^{h} q_{nz}\lambda\triangle\, dz\,\delta\vartheta_0 \Big)\Big\}\, ds \tag{3.11}$$

$$\delta A_4 = \oint_{(L)} \Big\{ \int_{-h}^{h} q_{nx} s\,dz\, \delta u_0' + \int_{-h}^{h} q_{ny} s\,dz\, \delta v_0' + \int_{-h}^{h} q_{nz}\, c\,dz\, \delta w_0 -$$

$$- \frac{m}{4(m-1)} \Big( \int_{-h}^{h} q_{nx}\, \lambda dz\, \delta\partial_1\vartheta_0' + \int_{-h}^{h} q_{ny}\lambda dz\, \delta\partial_2\vartheta_0' + \int_{-h}^{h} q_{nz} sz dz\, \delta\vartheta_0' \Big)\Big\}\, ds \tag{3.12}$$

In an actual calculation of the integrals in Expressions (3.11) and (3.12) we expand the loading $\mathbf{q}_n = \mathbf{q}_n(z)$ in a series in powers of $z$, then, after completing the relevant calculations the formulas for $\delta A_3$ and $\delta A_4$ are also represented in series in powers of the thickness of the plate.

Another method is to introduce the statical and hyperstatical characteristics of the distribution of external forces applied over the faces of the plate

$$R_x = \int_{-h}^{h} q_{nx} dz, \qquad R_y = \int_{-h}^{h} q_{ny} dz, \qquad Q = \int_{-h}^{h} q_{nz} dz$$

$$M_x = -\int_{-h}^{h} q_{ny} z dz, \qquad M_y = \int_{-h}^{h} q_{nx} z dz, \qquad W = \int_{-h}^{h} q_{nz} z dz \tag{3.13}$$

$$R_x^{(2n)} = \frac{(-1)^n}{(2n)!} \int_{-h}^{h} q_{nx} z^{2n}\, dz, \qquad R_y^{(2n)} = \frac{(-1)^n}{(2n)!} \int_{-h}^{h} q_{ny} z^{2n}\, dz$$

$$M_x^{(2n+1)} = -\frac{(-1)^n}{(2n+1)!}\int_{-h}^{h} q_{ny} z^{2n+1}\,dz, \qquad M_y^{(2n+1)} = \frac{(-1)^n}{(2n+1)!}\int_{-h}^{h} q_{nx} z^{2n+1}\,dz \quad (3.14)$$

$$Q^{(2n)} = \frac{(-1)^n}{(2n)!}\int_{-h}^{h} q_{nz} z^{2n}\,dz, \qquad W^{(2n+1)} = \frac{(-1)^n}{(2n+1)!}\int_{-h}^{h} q_{nz} z^{2n+1}\,dz$$

Here $R_x$ and $R_y$ are the projections of the principal vector of the surface loading on the $x$ and $y$ axes; $Q$ is the shear force; $M_x$ and $M_y$ are moments about the $x$ and $y$ axes. In addition we have the hyperstatical characteristics: $W$ is a biforce; $R_x^{(2n)}$ and $R_y^{(2n)}$ are polyvectors; $M_x^{(2n+1)}$, $M_y^{(2n+1)}$ are polymoments; $Q^{(2n)}$ are shear forces and finally, $W^{(2n+1)}$ are polyforces (polybiforces) of various orders. Letting $\sigma$, $s$, $\lambda$ assume their original meaning (0.3) and using notations (3.13) and (3.14), we obtain

$$\delta A_3 = \oint_{(L)}\Big\{R_x\delta u_0 + R_y\delta v_0 + W\delta w_0' + \sum_{n=1}^{\infty}\Big[R_x^{(2n)}\cdot\delta\left(\triangle^n u_0 + \frac{nm}{m-2}\,\partial_1\triangle^{n+1}\vartheta_0\right) +$$

$$+ R_y^{(2n)}\,\delta\left(\triangle^n v_0 + \frac{nm}{m-2}\,\partial_2\triangle^{n-1}\vartheta_0\right) + W^{(2n+1)}\,\delta\left(\triangle^n w_0' - \frac{nm}{m-2}\triangle^n\vartheta_0\right)\Big]\Big\}\,ds \quad (3.15)$$

$$\delta A_4 = \oint_{(L)}\{M_y\cdot\delta u_0' - M_x\cdot\delta v_0' + Q\delta w_0 + \sum_{n=1}^{\infty}\Big[M_y^{(2n+1)}\cdot\delta\left(\triangle^n u_0' + \frac{nm}{2m-2}\,\partial_1\triangle^{n-1}\vartheta_0'\right) -$$

$$- M_x^{(2n+1)}\delta\left(\triangle^n v_0' + \frac{nm}{2m-2}\,\partial_2\triangle^{n-1}\vartheta_0'\right) + Q^{(2n)}\cdot\delta\left(\triangle^n w_0 + \frac{nm}{2m-2}\triangle^{n-1}\vartheta_0'\right)\Big]\Big\}\,ds \quad (3.16)$$

**4. Differential equations of the theory of thick plates.** In order to derive the differential equations (and boundary conditions) we equate to zero the variation of potential energy of the whole system which contains the variation of potential strain energy of the plate and the variation of potential energy of the external forces. The latter is equal to the elemental work done by these forces but with a negative sign. The extension-compression and bending problems can be studied separately. From the principle of minimum potential energy for both these problems we have

$$\delta\Pi_1 - \delta A_1 - \delta A_3 = 0, \qquad \delta\Pi_2 - \delta A_2 - \delta A_4 = 0 \quad (4.1)$$

The quantities which occur here are given by Expressions (1.25),(3.7), (3.14),(2.14),(3.8) and (3.15). Formulas (4.1) contain double integrals over the region of the plate $(\Omega)$ and an infinite series of line integrals. Equating to zero, the coefficients of the independent variations $\delta u_0$, $\delta v_0$, $\delta w_0'$, $\delta u_0'$, $\delta v_0'$ and $\delta w_0$ in the double integrals, we obtain the differential equations for the problem of extension $L$ and of bending $K$

$$L^{(1)} = \frac{1}{\mu}(C\eta_x - \partial_1\Xi), \qquad K^{(1)} = \frac{1}{\mu}(St_x - \partial_1\Theta)$$

$$L^{(2)} = \frac{1}{\mu}(C\eta_y - \partial_2\Xi), \qquad K^{(2)} = \frac{1}{\mu}(St_y - \partial_2\Theta) \quad (4.2)$$

$$-L^{(0)} = \frac{1}{\mu}(-S\zeta - \Xi), \qquad -K^{(0)} = \frac{1}{\mu}(-Cp + \Delta\Theta)$$

In the last of the equations of system (4.2) the signs have been changed.

The differential equations of equilibrium for a slab have been obtained by a different method by Lur'e [2]. We introduce the column matrices (vectors)

$$u = \begin{Vmatrix} u_0 \\ v_0 \\ w_0' \end{Vmatrix}, \qquad w = \begin{Vmatrix} u_0' \\ v_0' \\ w_0 \end{Vmatrix}, \qquad \eta = \begin{Vmatrix} \eta_x \\ \eta_y \\ -\zeta \end{Vmatrix}, \qquad t = \begin{Vmatrix} t_x \\ t_y \\ -p \end{Vmatrix} \tag{4.3}$$

and the square matrices $a = \| a_{kl} \|$, $\alpha = \| \alpha_{kl} \|$ with elements

$$a_{kk} = -2\left(S\triangle + \frac{mhC}{m-2}\partial_k^2\right), \quad a_{kl} = -\frac{2mhC}{m-2}\partial_k\partial_l, \qquad a_{k3} = 2\partial_k\left(S - \frac{mhC}{m-2}\right)$$

$$a_{3k} = -\frac{2\partial_k}{m-2}(2C + mhS\triangle), \qquad a_{33} = -\frac{2}{m-2}[2(m-1)C + mhS\triangle] \tag{4.4}$$

$$\alpha_{kk} = 2C - \frac{mhS}{m-1}\partial_k^2, \qquad \alpha_{kl} = -\frac{mhS}{m-1}\partial_k\partial_l, \qquad \alpha_{k3} = \partial_k\left(2C + \frac{mhS\triangle}{m-1}\right)$$

$$\alpha_{3k} = \frac{\partial_k}{m-1}[(m-2)S + mhC], \qquad \alpha_{33} = \frac{\triangle}{m-1}[(3m-2)S - mhC] \tag{4.5}$$

where $k,\ l = 1,\ 2$. Lur'e's equations can then be written in the form

$$au = \frac{1}{\mu}\eta, \qquad \alpha w = \frac{1}{\mu}t \tag{4.6}$$

The matrices $a$ and $\alpha$ transform the vectors $u$, $w$ (to the accuracy of the factor $^1/\mu$) into stress vectors on the faces of the plate. The determinants of the matrices $a$ and $\alpha$ give the operators of the solving equations for the stress functions of Lur'e [2 and 4]. These determinants are

$$|a| = -\frac{16m}{m-2}(CS+h)S\triangle^2, \qquad |\alpha| = \frac{8m}{m-1}(CS-h)C\triangle \tag{4.7}$$

In addition to the matrices $a$ and $\alpha$, we introduce the matrices $b$ and $\beta$ which transform the vectors $u$ and $w$ into the displacements at the faces of the plate. The elements of these matrices are as follows:

$$b_{kk} = C - \frac{mhS\partial_k^2}{2(m-2)}, \qquad b_{kl} = -\frac{mhS\partial_k\partial_l}{2(m-2)}, \qquad b_{k3} = -\frac{mhS\partial_k}{2(m-2)}$$

$$b_{3k} = \frac{m\Lambda\triangle\partial_k}{2(m-2)}, \qquad b_{33} = S + \frac{m\Lambda\triangle}{2(m-2)} \tag{4.8}$$

$$\beta_{kk} = S - \frac{m\Lambda\partial_k^2}{4(m-1)}, \qquad \beta_{kl} = -\frac{m\Lambda\partial_k\partial_l}{4(m-1)}, \qquad \beta_{k3} = \frac{m\Lambda\triangle\partial_k}{4(m-1)}$$

$$\beta_{3k} = -\frac{mhS\partial_k}{4(m-1)}, \qquad \beta_{33} = C + \frac{mhS\triangle}{4(m-1)} \tag{4.9}$$

The determinants of the matrices $b$ and $\beta$ give the operators of the solving equations for the functions of displacements in problems of a slab or a plate for displacements specified on the faces $z = \pm h$. These determinants have the following values:

$$|b| = \frac{C}{2(m-2)}[(3m-4)CS - mh], \qquad |\beta| = \frac{S}{4(m-1)}[(3m-4)CS + mh] \tag{4.10}$$

Equations (4.2) can be derived from Lur'e's equations (4.6) by pre-multiplying them by the transpose matrices $b^*$ and $\beta^*$, the matrices so obtained

$$e = b^*a, \qquad \varepsilon = \beta^*\alpha \tag{4.11}$$

in contrast to the matrices $a$, $\alpha$, $b$, $\beta$, are symmetrical. Their elements will not be written out since they form the left-hand sides of Equations (4.2) defined by (1.24) and (2.22). The determinants of the matrices $e$ and $\varepsilon$ can be obtained by multiplying the first of the determinants (4.7) and

(4.10) or the second of the determinants (4.7) by (4.10). Since the matrices $b$ and $\beta$ are nondegenerate (*), there is complete agreement between Lur'e's equations (4.6) and Equations (4.9) derived here. From now on we shall start from Lur'e's equations, since these are simpler.

Assuming the operators $C$ and $S$ in (4.4) and (4.5) to have their initial meaning as series in powers of the thickness of the plate, we can rewrite the system (4.6) in the form

$$\begin{gathered}\sum_{n=0}^{\infty}\frac{(-1)^n h^{2n+1}}{(2n+1)!}\triangle^n\left[\partial_1 w_0' - \triangle u_0 - (2n+1)\frac{m\partial_1\vartheta_0}{m-2}\right] = \frac{\eta_x}{2\mu}\\ \sum_{n=0}^{\infty}\frac{(-1)^n h^{2n}}{(2n)!}\triangle^n\left[\frac{2nm-1}{m-1}\vartheta_0 - w_0'\right] = -\frac{\zeta}{4\mu}\end{gathered}\qquad(4.12)$$

$$\begin{gathered}\sum_{n=0}^{\infty}\frac{(-1)^n h^{2n}}{(2n)!}\triangle^n\left[\partial_1 w_0 + u_0' + \frac{nm\partial_1\vartheta_0'}{(m-1)\triangle}\right] = \frac{t_x}{2\mu}\\ \sum_{n=0}^{\infty}\frac{(-1)^n h^{2n+1}}{(2n+1)!}\triangle^n\left[\triangle w_0 + \frac{nm+m-1}{2(m-1)}\vartheta_0'\right] = -\frac{p}{4\mu}\end{gathered}\qquad(4.13)$$

We have omitted the second equations of systems (4.12) and (4.13); they can be derived from the first equations of these systems by replacing $\partial_1, u_0, \eta_x, u_0'$ and $t_x$ by $\partial_2, v_0, \eta_y, v_0'$ and $t_y$.

From equations (4.12) with $n = 0$ and in the absence of surface loading, we can derive the familiar equations of the plane problem expressed in terms of the displacements $u_0$ and $v_0$ ($w_0'$ is easily eliminated). The next approximation ($n = 0$ and $n = 1$) refines the plane problem and leads to a fifth order harmonic system of differential equations requiring five boundary conditions. For system (4.13) retention of only the first terms ($n = 0$) gives no result (**). Retaining two terms of the series in each of Equations (4.13) we obtain, after eliminating the rotations $u_0'$, and $v_0'$ the biharmonic equation of the theory of thin plates. In order to obtain a more accurate theory of bending more terms must be retained in the summations in (4.13).

**5. Boundary conditions of the theory of thick plates.** Relations (4.1) lead to boundary conditions, as well as differential equations, in the form of an infinite series of line integrals. Confining our attention to a specific power of $h$, we can derive the boundary conditions for a system of differential equations corresponding to the approximation selected. There are in addition certain general concepts worthy of mention in connection with the boundary conditions for thick plate theory.

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*) Degeneration of the matrices $b$ and $\beta$ occurs when there is no displacement at $z = \pm h$, i.e. when the faces of the plate are fixed. This case is not considered here.

**) All the displacements ($u_0'$, $v_0'$, $w_0$) are eliminated from the system leaving the condition of equilibrium of external forces $\partial_1 t_x + \partial_2 t_y + p\,h = 0$

Formulas (3.13) to (3.16) lead to the obvious requirements applicable to the force conditions on the edge of a thick plate: the equality of the statical characteristics (the principal vector and principal moment) of the external surface forces to the statical characteristics of the stresses $\sigma_n$, $\tau_{ns}$, $\tau_{nz}$ as well as the hyperstatical equivalence characterized by biforces, polyvectors, polymoments and shear forces of higher orders. In particular, on an unloaded free edge of the plate the integral characteristics (3.13) and (3.14) of the stresses $\sigma_n$, $\tau_{ns}$ and $\tau_{nz}$ must be equated to zero.

Similarly, for a built-in edge of the plate (in the extensional problem) we must impose the conditions

$$\begin{gathered} u_0 = 0, \quad v_0 = 0, \quad w_0' = 0, \quad \triangle^n u_0 + \frac{mn}{m-2}\partial_1 \triangle^{n-1}\vartheta_0 = 0 \\ \triangle^n v_0 + \frac{nm}{m-2}\partial_2 \triangle^{n-1}\vartheta_0 = 0, \ \triangle^n w_0' - \frac{nm}{m-2}\triangle^n \vartheta_0 = 0 \qquad (n = 1, 2, \ldots) \end{gathered} \tag{5.1}$$

since if the force factors are nonzero the variations of these quantities in the expression for the elemental work (3.15) vanish.

In the problem of bending of a plate the conditions for a built-in edge are

$$\begin{gathered} u_0' = 0, \quad v_0' = 0, \quad w_0 = 0, \ \triangle^n u_0' + \frac{nm}{2(m-1)}\partial_1 \triangle^{n-1}\vartheta_0' = 0 \\ \triangle^n v_0' + \frac{nm}{2(m-1)}\partial_2 \triangle^{n-1}\vartheta_0' = 0, \ \triangle^n w_0 + \frac{nm}{2(m-1)}\triangle^{n-1}\vartheta_0' = 0 \quad (n = 1, 2, \ldots) \end{gathered} \tag{5.2}$$

Note also that in est blishing finally the boundary conditions for any particular approximation (the series of line integrals are terminated at a specific power of $h$ ) it is essential to take into account also the equations relating the variations $\delta u_0$, $\delta v_0$, $\delta w_0'$, $\delta u_0'$, $\delta v_0'$, $\delta w_0$ and their derivatives. These are obtained by varying Equations (4.12) and (4.13) shortened to a specific power of $h$. If we take into account quantities of the first order relative to $h$ in the line integrals (1.25) and (3.15) (*) which appear in the first of relations (4.1), we arrive at the boundary conditions for the plane problem.

$$\begin{gathered} \oint_{(L)} \Big\{ \Big[ 4\mu h \Big( \frac{\partial u_{n0}}{\partial n} - u_{s0}\frac{\partial \alpha}{\partial n} + \frac{\vartheta_0}{m-2} \Big) - R_n \Big] \cdot \delta u_{n0} + \\ + \Big[ 2\mu h \Big( \frac{\partial u_{n0}}{\partial s} + u_{n0}\frac{\partial \alpha}{\partial n} + \frac{\partial u_{s0}}{\partial n} - u_{s0}\frac{\partial \alpha}{\partial s} \Big) - R_s \Big] \cdot \delta u_{s0} \Big\} ds = 0 \end{gathered} \tag{5.3}$$

Here $R_s$ and $R_n$ are the projections of the principal vector of the edge forces on to the tangent and normal to the contour, $u_{s0}$ and $u_{s0}$ are determined from (1.11) and $\alpha$ denotes the angle between the normal and the $x$-axis.

In order to derive the boundary conditions for the theory of thin plates, we retain in the line integrals (2.14) terms of the order of $h$ and $h^3$, then, marking these integrals with an asterisk, we have

*) The comparable order of the integral characteristics of the loading on the face of the plate can be easily seen from Formulas (3.13) and (3.14).

$$\frac{\delta\Pi_2^*}{\mu} = 2h\oint_{(L)}\left(n_x u_0' + n_y v_0' + \frac{\partial w_0}{\partial n}\right)\cdot\delta w_0 ds + \frac{2h^3}{3}\oint_{(L)}\left\{2\frac{\partial u_0'}{\partial n}\cdot\delta u_0' + 2\frac{\partial v_0'}{\partial n}\cdot\delta v_0' + \right.$$
$$+ \frac{2m\Delta w_0 + (m+1)\vartheta_0'}{m-1}\cdot\delta u_{n0}' + (\partial_2 u_0' - \partial_1 v_0')\cdot\delta u_{s0}' - \tag{5.4}$$
$$-\frac{m}{2(m-1)}\left(n_x u_0' + n_y v_0' + \frac{\partial w_0}{\partial n}\right)\cdot\delta\vartheta_0' - \frac{(\vartheta_0' + 2\triangle w_0)}{m-1}\cdot\delta\left(\frac{\partial w_0}{\partial n}\right) -$$
$$\left. -\left[n_x\triangle u_0' + n_y\triangle v_0' + \frac{m-3}{m-1}\cdot\frac{\partial\triangle w_0}{\partial n} + \frac{m-2}{2(m-1)}\frac{\partial\vartheta_0'}{\partial n}\right]\delta w_0\right\} ds$$

But from the equilibrium equations (4.13) for the order of accuracy indicated (for simplicity we take $t_x = t_y = 0$ ) it follows that

$$n_x u_0' + n_y v_0' + \frac{\partial w_0}{\partial n} = \frac{h^2}{2}\left(\frac{\partial\triangle w_0}{\partial n} + n_x\triangle u_0' + n_y\triangle v_0' + \frac{m}{m-1}\frac{\partial\vartheta_0'}{\partial n}\right)$$
$$u_0' + \partial_1 w_0 = O(h^2), \qquad v_0' + \partial_2 w_0 = O(h^2), \qquad \vartheta_0' + 2\triangle w_0 = O(h^2) \tag{5.5}$$

Substituting (5.5) into (5.4) leads to Expression

$$\delta\Pi_2^* = \frac{4}{3}\mu h^3\oint_{(L)}\left\{\left[\frac{\partial}{\partial n}\left(\frac{\partial w_0}{\partial s}\right) + \frac{\partial w_0}{\partial n}\frac{\partial\alpha}{\partial n}\right]\cdot\delta\left(\frac{\partial w_0}{\partial s}\right) - \right.$$
$$\left. -\frac{m}{m-1}\cdot\frac{\partial\triangle w_0}{\partial n}\cdot\delta w_0 + \left[\frac{\triangle w_0}{m-1} + \frac{\partial^2 w_0}{\partial n^2} - \frac{\partial w_0}{\partial s}\cdot\frac{\partial\alpha}{\partial n}\right]\cdot\delta\left(\frac{\partial w_0}{\partial n}\right)\right\} ds \tag{5.6}$$

We now introduce the bending and twisting moments of the edge of the plate

$$G = M_s = n_x M_y - n_y M_x, \qquad H = -M_n = -n_x M_x - n_y M_y$$

as well as Poisson's ratio $\nu = 1/m$ and the plate stiffness

$$D = \frac{4\mu h^3 m}{3(m-1)}$$

Then, substituting (5.6) and (3.16) into the second relation of (4.1) and taking into account

$$\triangle w_0 = \frac{\partial^2 w_0}{\partial n^2} + \frac{\partial^2 w_0}{\partial s^2} + \frac{\partial w_0}{\partial n}\frac{\partial\alpha}{\partial s} - \frac{\partial w_0}{\partial s}\frac{\partial\alpha}{\partial n}$$

we obtain the boundary conditions for the theory of thin plates

$$\oint_{(L)}\left\{\left[D(1-\nu)\left(\frac{\partial}{\partial n}\left(\frac{\partial w_0}{\partial s}\right) + \frac{\partial w_0}{\partial n}\frac{\partial\alpha}{\partial n}\right) + H\right]\cdot\delta\left(\frac{\partial w_0}{\partial s}\right) - \left(D\frac{\partial\triangle w_0}{\partial n} + Q\right)\delta w_0 + \right.$$
$$\left. + \left[D\left(\frac{\partial^2 w_0}{\partial n^2} - \frac{\partial w_0}{\partial s}\frac{\partial\alpha}{\partial n} + \nu\left(\frac{\partial^2 w_0}{\partial s^2} + \frac{\partial w_0}{\partial n}\frac{\partial\alpha}{\partial s}\right)\right) + G\right]\delta\left(\frac{\partial w_0}{\partial n}\right)\right\} ds = 0 \tag{5.7}$$

Thus we have obtained Poisson's boundary conditions. By the usual method integrating the terms containing $\delta(\partial w_0/\partial s)$ by parts we can also obtain Kirchhoff's boundary conditions from (5.7).

## BIBLIOGRAPHY

1. Lur'e, A.I., K zadache o ravnovesii plastiny peremennoi tolshchiny (On the problem of the equilibrium of a plate of variable thickness). Trudy leningr.ind.Inst., № 6, p.57, 1936.

2. Lur'e, A.I., K teorii tolstykh plit (On the theory of thick plates). *PMM* Vol.6, №№ 2 and 3, 1942.

3. Prokopov, V.K., Obobshchenie formuly Grina (Generalization of Green's formula). *PMM* Vol.28, № 1, 1964.

4. Lur'e, A.I., Prostranstvennye zadachi teorii uprugosti (Three-dimensional Problems of the Theory of Elasticity). Gostekhizdat, 1955.

5. Aksentian, O.K. and Vorivich, I.I., Napriazhennoe sostoianie plity maloi tolshchiny (State of stress in a plate of small thickness). *PMM* Vol. 27, № 6, 1963.


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